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Twisted tensor products of nonlocal vertex algebras

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ABSTRACT

In this paper we introduce and study a twisted tensor product construction of nonlocal vertex algebras. Among the main results, we establish a universal property and give a characterization of a twisted tensor product. Furthermore, we give a construction of modules for a twisted tensor product. We also show that smash products studied by one of us before can be realized as twisted tensor products.

1. Introduction

For associative algebras, there is a notion of twisted tensor product, generalizing the notion of tensor product. Let A and B be two associative unital algebras. A twisted tensor product $A \otimes_R B$ (see [CSV,VV]) is associated to each twisting operator R in the sense that $R : B \otimes A \to A \otimes B$ is a linear map satisfying

$$R(1 \otimes a) = a \otimes 1, \qquad R(b \otimes 1) = 1 \otimes b \quad \text{for } a \in A,\ b \in B,$$
$$R(\mu_B \otimes 1) = (1 \otimes \mu_B)R_{12}R_{23}, \qquad R(1 \otimes \mu_A) = (\mu_A \otimes 1)R_{23}R_{12},$$

where μ_A and μ_B denote the multiplications of A and B. The multiplication μ of $A \otimes_R B$ is given by

$$\mu\big(a \otimes b \otimes a' \otimes b'\big) = (\mu_A \otimes \mu_B)\big(a \otimes R\big(b \otimes a'\big) \otimes b'\big)$$

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for $a, a' \in A$, $b, b' \in B$. Twisted tensor product, as well as its generalizations, provides an effective tool to construct new algebras for various purposes (cf. [PPO]).

Vertex algebras are both analogues and generalizations of commutative and associative unital algebras, while nonlocal vertex algebras (or field algebras in the sense of [BK]) are analogues and generalizations of associative unital algebras. In [EK], one of an important series of papers, Etingof and Kazhdan developed a fundamental theory of quantum vertex operator algebras in the sense of formal deformation, where quantum vertex operator algebras are ($\hbar$-adic) nonlocal vertex algebras (over $\mathbb{C}[[\hbar]]$) which satisfy what was called $\mathcal{S}$-locality. Partly motivated by this, we developed a theory of (weak) quantum vertex algebras (see [L3,L4]), where weak quantum vertex algebras are generalizations of vertex superalgebras, instead of formal deformations. Weak quantum vertex algebras in this sense are nonlocal vertex algebras that satisfy a variation of Etingof–Kazhdan's $\mathcal{S}$-locality. In this developing theory, constructing interesting examples of quantum vertex algebras is one of the most important problems.

In this paper, we study twisted tensor products of nonlocal vertex algebras and of (weak) quantum vertex algebras. The main purpose is to build various tools for constructing new interesting quantum vertex algebras. Let U and V be two nonlocal vertex algebras. We define a twisting operator to be a linear map

$$R(x) : V \otimes U \to U \otimes V \otimes \mathbb{C}((x)),$$

satisfying a set of conditions which are stringy analogues of those listed before for a twisting operator with associative algebras. The underlying space of the twisted tensor product $U \otimes_R V$ associated to R is $U \otimes V$, while the vacuum vector is $\mathbf{1}_U \otimes \mathbf{1}_V$ and the vertex operator map, denoted by Y_R, is given by

$$Y_R(u \otimes v, x)\big(u' \otimes v'\big) = \big(Y_U(x) \otimes Y_V(x)\big)\big(u \otimes R(x)\big(v \otimes u'\big) \otimes v'\big)$$

for $u, u' \in U$, $v, v' \in V$. It is proved that $U \otimes_R V$ is a nonlocal vertex algebra, containing U and V canonically as subalgebras which satisfy a certain commutation relation. (If both U and V are weak quantum vertex algebras, it is proved that $U \otimes_R V$ is a weak quantum vertex algebra.) On the other hand, it is proved that if a nonlocal vertex algebra K, which is non-degenerate in the sense of [EK], contains subalgebras U and V satisfying a certain commutation relation, then K is isomorphic to the twisted tensor product $U \otimes_R V$ with respect to a twisting operator $R(x)$. Also established in this paper is a universal property for the twisted tensor product $U \otimes_R V$, similar to the one for the ordinary tensor product $U \otimes V$. Regarding $U \otimes_R V$-modules, it is proved that a U-module structure and a V-module structure compatible in a certain sense on a vector space W give rise to a $U \otimes_R V$-module structure canonically.

In [L5], a notion of nonlocal vertex bialgebra and a notion of nonlocal vertex module-algebra for a nonlocal vertex bialgebra were formulated, and a smash product construction of nonlocal vertex algebras was established. Given a nonlocal vertex bialgebra H and for a nonlocal vertex H-module algebra U, we have a smash product $U\sharp H$. In the present paper, we slightly generalize the smash product construction with H replaced by what we call a nonlocal vertex H-comodule-algebra, and we show that the smash product $U\sharp V$ is a twisted tensor product with respect to a canonical twisting operator.

This paper was partly motivated by a recent study [LS] on regular representations for what we called Möbius quantum vertex algebras. Previously, a theory of regular representations for a general vertex operator algebra V was developed in [L1], where the regular representation space was proved to have a canonical module structure for the ordinary tensor product $V \otimes V$. Furthermore, under suitable assumptions on V, a result of Peter–Weyl type was obtained. In [LS] we extended this theory for a Möbius quantum vertex algebra V, and we proved that the regular representation space has a canonical module structure for a certain twisted tensor product of V with V, instead of the ordinary tensor product. This motivated us to study more general twisted tensor products of nonlocal vertex algebras.

We mention that Anguelova and Bergvelt studied a notion of what they called H_D-quantum vertex algebra in [AB] and they gave a construction by employing Borcherds' bicharacter construction (see [Bo]).

This paper is organized as follows: In Section 2, we present some basic notions and twisted tensor product. In Section 3, we show that twisted tensor products are (weak) quantum vertex algebras if the tensor factors are (weak) quantum vertex algebras. In Section 4, we show that smash products of nonlocal vertex algebras over a nonlocal vertex bialgebra are twisted tensor products.

2. Twisted tensor product nonlocal vertex algebras

In this section, first we define the notion of twisting operator and construct the twisted tensor product nonlocal vertex algebra. Then we establish a universal property and give a characterization of the twisted tensor product. We also present an existence theorem for a module structure for the twisted tensor product.

We begin by recalling the notion of nonlocal vertex algebra. A *nonlocal vertex algebra* (see [L2], cf. [BK]) is a vector space V, equipped with a linear map

$$Y(\cdot,x): V \to \operatorname{Hom}\big(V, V((x))\big) \subset (\operatorname{End} V)\big[\big[x, x^{-1}\big]\big]$$
$$v \mapsto Y(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \operatorname{End} V),$$

and equipped with a vector $\mathbf{1} \in V$, satisfying the conditions that for $v \in V$,

$$Y(\mathbf{1},x)v = v, \qquad Y(v,x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x\to 0} Y(v,x)\mathbf{1} = v, \tag{2.1}$$

and that for $u, v, w \in V$, there exists a nonnegative integer k such that

$$(x_0+x_2)^k Y(u, x_0+x_2)Y(v,x_2)w = (x_0+x_2)^k Y\big(Y(u,x_0)v, x_2\big)w \tag{2.2}$$

(*weak associativity*).

We sometimes denote a nonlocal vertex algebra by a triple $(V, Y, \mathbf{1})$, to emphasize the *vertex operator map* Y and the *vacuum vector* $\mathbf{1}$.

Let V be a nonlocal vertex algebra. Define a linear operator $\mathcal{D}$ on V by

$$\mathcal{D}(v) = v_{-2}\mathbf{1} \quad \text{for } v \in V. \tag{2.3}$$

Then

$$\big[\mathcal{D}, Y(v,x)\big] = Y(\mathcal{D}v, x) = \frac{d}{dx}Y(v,x) \quad \text{for } v \in V. \tag{2.4}$$

We also have

$$Y(v,x)\mathbf{1} = e^{x\mathcal{D}}v = e^{x\mathcal{D}}Y(\mathbf{1},-x)v. \tag{2.5}$$

For a nonlocal vertex algebra V, a V-*module* is a vector space W, equipped with a linear map

$$Y_W(\cdot,x): V \to \operatorname{Hom}\big(W, W((x))\big) \subset (\operatorname{End} W)\big[\big[x, x^{-1}\big]\big],$$

satisfying the conditions that $Y_W(\mathbf{1}, x) = 1_W$ (the identity operator on W) and that for $u, v \in V$, $w \in W$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_0 + x_2)^l Y_W\big(Y(u, x_0)v, x_2\big) w.$$

Let (W, Y_W) be a V-module. Note that for $u, v \in V$,

$$\big(Y_W(u, x_1) Y_W(v, x_2)\big)\big|_{x_1 = x_0 + x_2} = Y_W(u, x_0 + x_2) Y_W(v, x_2),$$

which by the formal Taylor theorem equals $e^{x_2 \frac{\partial}{\partial x_0}}(Y_W(u, x_0) Y_W(v, x_2))$, exists, but $(Y_W(u, x_1) Y_W(v, x_2))|_{x_1 = x_2 + x_0}$ in general does *not* exist. On the other hand, if $A(x_1, x_2)$ is an element of $\mathrm{Hom}(W, W((x_1, x_2)))$, then both $A(x_1, x_2)|_{x_1 = x_2 + x_0}$ and $A(x_1, x_2)|_{x_1 = x_0 + x_2}$ exist.

As we shall need, we recall another form of weak associativity from [LTW].

Lemma 2.1. *Let V be a nonlocal vertex algebra. In the definition of a V-module, the weak associativity axiom can be equivalently replaced with the property that for $u, v \in V$, there exists a nonnegative integer k such that*

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \mathrm{Hom}\big(W, W((x_1, x_2))\big) \tag{2.6}$$

and

$$\big((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)\big)\big|_{x_1 = x_2 + x_0} = x_0^k Y_W\big(Y(u, x_0)v, x_2\big). \tag{2.7}$$

Furthermore, for a V-module W and for any $u, v \in V$, (2.7) *holds for every nonnegative integer k satisfying* (2.6)*.*

Proof. The first part is a special case of Lemma 2.9 of [LTW] with $\sigma = 1$. As for the second part, suppose that k is a nonnegative integer such that both (2.6) and (2.7) hold, and let k' be any nonnegative integer such that (2.6) holds. Note that

$$\big((x_1 - x_2)^r Y_W(u, x_1) Y_W(v, x_2)\big)\big|_{x_1 = x_2 + x_0}$$

for $r = k$ and for $r = k'$ both exist in $\mathrm{Hom}(W, ((x_2))[[x_0]])$. Then

$$\begin{aligned}
&x_0^k \big((x_1 - x_2)^{k'} Y_W(u, x_1) Y_W(v, x_2)\big)\big|_{x_1 = x_2 + x_0} \\
&\quad = \big((x_1 - x_2)^k\big)\big|_{x_1 = x_2 + x_0} \cdot \big((x_1 - x_2)^{k'} Y_W(u, x_1) Y_W(v, x_2)\big)\big|_{x_1 = x_2 + x_0} \\
&\quad = \big((x_1 - x_2)^{k + k'} Y_W(u, x_1) Y_W(v, x_2)\big)\big|_{x_1 = x_2 + x_0} \\
&\quad = \big((x_1 - x_2)^{k'}\big)\big|_{x_1 = x_2 + x_0} \cdot \big((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)\big)\big|_{x_1 = x_2 + x_0} \\
&\quad = x_0^{k + k'} Y_W\big(Y(u, x_0)v, x_2\big),
\end{aligned}$$

from which we immediately get

$$\big((x_1 - x_2)^{k'} Y_W(u, x_1) Y_W(v, x_2)\big)\big|_{x_1 = x_2 + x_0} = x_0^{k'} Y_W\big(Y(u, x_0)v, x_2\big),$$

as desired. □

For a nonlocal vertex algebra $(V, Y, \mathbf{1})$, we follow [EK] to define a linear map

$$Y(x) : V \otimes V \to V((x))$$

by

$$Y(x)(u \otimes v) = Y(u, x)v \quad \text{for } u, v \in V.$$

Similarly, for a V-module (W, Y_W), we denote by $Y_W(x)$ the associated linear map

$$Y_W(x) : V \otimes W \to W((x)).$$

Let U and V be two nonlocal vertex algebras. We have an (ordinary) tensor product nonlocal vertex algebra $U \otimes V$, where the vacuum vector is $\mathbf{1} \otimes \mathbf{1}$ and the vertex operator map is given by

$$Y(u \otimes v, x)\big(u' \otimes v'\big) = Y(u, x)u' \otimes Y(v, x)v' \quad \text{for } u, u' \in U,\ v, v' \in V.$$

That is,

$$Y_{U \otimes V}(x) = \big(Y_U(x) \otimes Y_V(x)\big)\sigma^{23},$$

where σ^{23} is the linear operator on $(U \otimes V)^{\otimes 2}$, defined by

$$\sigma^{23}\big(u \otimes v \otimes u' \otimes v'\big) = u \otimes u' \otimes v \otimes v'$$

for $u, u' \in U,\ v, v' \in V$.

Definition 2.2. Let U and V be nonlocal vertex algebras. A *twisting operator* for the ordered pair (U, V) is a linear map

$$R(x) : V \otimes U \to U \otimes V \otimes \mathbb{C}((x)),$$

satisfying the following conditions:

$$R(x)(v \otimes \mathbf{1}) = \mathbf{1} \otimes v \quad \text{for } v \in V, \tag{2.8}$$

$$R(x)(\mathbf{1} \otimes u) = u \otimes \mathbf{1} \quad \text{for } u \in U, \tag{2.9}$$

$$R(x_1)\big(1 \otimes Y(x_2)\big) = \big(Y(x_2) \otimes 1\big)R^{23}(x_1)R^{12}(x_1 + x_2), \tag{2.10}$$

$$R(x_1)\big(Y(x_2) \otimes 1\big) = \big(1 \otimes Y(x_2)\big)R^{12}(x_1 - x_2)R^{23}(x_1). \tag{2.11}$$

We say that a twisting operator $R(x)$ is *invertible* if $R(x)$, viewed as a $\mathbb{C}((x))$-linear map from $V \otimes U \otimes \mathbb{C}((x))$ to $U \otimes V \otimes \mathbb{C}((x))$, is invertible. The inverse of an invertible $R(x)$ is a $\mathbb{C}((x))$-linear map $R^{-1}(x)$ from $U \otimes V \otimes \mathbb{C}((x))$ to $V \otimes U \otimes \mathbb{C}((x))$. We often consider $R^{-1}(x)$ as a $\mathbb{C}$-linear map

$$R^{-1}(x) : U \otimes V \to V \otimes U \otimes \mathbb{C}((x)).$$

The following is straightforward:

Lemma 2.3. *If $R(x)$ is an invertible twisting operator for the ordered pair (U,V), then $R^{-1}(-x)$ is an invertible twisting operator for the ordered pair (V,U).*

We now present the twisted tensor product.

Theorem 2.4. *Let U, V be nonlocal vertex algebras and let $R(x)$ be a twisting operator of the ordered pair (U,V). Set*

$$Y_R(x) = \big(Y(x)\otimes Y(x)\big)R^{23}(-x). \tag{2.12}$$

Then $(U\otimes V, Y_R, \mathbf{1}\otimes\mathbf{1})$ carries the structure of a nonlocal vertex algebra, which contains U and V canonically as nonlocal vertex subalgebras.

Proof. For $u,u'\in U$, $v,v'\in V$, by definition we have

$$Y_R(x)\big(u\otimes v\otimes u'\otimes v'\big) = \sum_{i=1}^{r} f_i(-x)Y(u,x)u'^{(i)}\otimes Y\big(v^{(i)},x\big)v', \tag{2.13}$$

where

$$R(x)\big(v\otimes u'\big) = \sum_{i=1}^{r} u'^{(i)}\otimes v^{(i)}\otimes f_i(x)\in U\otimes V\otimes \mathbb{C}((x)).$$

As

$$f_i(-x)\in\mathbb{C}((x)),\qquad Y(u,x)u'^{(i)}\in U((x)),\qquad Y\big(v^{(i)},x\big)v'\in V((x))$$

for $1\leqslant i\leqslant r$, we see that $Y_R(u\otimes v,x)(u'\otimes v')$ exists in $(U\otimes V)((x))$.

For $u\in U$, $v\in V$, with $R(x)(\mathbf{1}\otimes u)=u\otimes\mathbf{1}$, we have

$$\begin{aligned} Y_R(\mathbf{1}\otimes\mathbf{1},x)(u\otimes v) &= \big(Y(x)\otimes Y(x)\big)(\mathbf{1}\otimes u\otimes\mathbf{1}\otimes v)\\ &= Y(\mathbf{1},x)u\otimes Y(\mathbf{1},x)v\\ &= u\otimes v. \end{aligned}$$

On the other hand, for $u\in U$, $v\in V$, with $R(x)(v\otimes\mathbf{1})=\mathbf{1}\otimes v$, we have

$$\begin{aligned} Y_R(u\otimes v,x)(\mathbf{1}\otimes\mathbf{1}) &= \big(Y(x)\otimes Y(x)\big)R^{23}(-x)(u\otimes v\otimes\mathbf{1}\otimes\mathbf{1})\\ &= \big(Y(x)\otimes Y(x)\big)(u\otimes\mathbf{1}\otimes v\otimes\mathbf{1})\\ &= Y(u,x)\mathbf{1}\otimes Y(v,x)\mathbf{1}, \end{aligned} \tag{2.14}$$

which lies in $(U\otimes V)[[x]]$, and

$$\lim_{x\to 0} Y_R(u\otimes v,x)(\mathbf{1}\otimes\mathbf{1}) = \lim_{x\to 0} Y(u,x)\mathbf{1}\otimes Y(v,x)\mathbf{1} = u\otimes v.$$

To see weak associativity, let $u,u',u''\in U$, $v,v',v''\in V$. Using (2.10) we get

$$\begin{aligned}
&Y_R(u\otimes v,x_0+x_2)Y_R\big(u'\otimes v',x_2\big)\big(u''\otimes v''\big)\\
&\quad=\big(Y(x_0+x_2)\otimes Y(x_0+x_2)\big)R^{23}(-x_0-x_2)\big(1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\big)\\
&\qquad\cdot R^{45}(-x_2)\big(u\otimes v\otimes u'\otimes v'\otimes u''\otimes v''\big)\\
&\quad=\big(Y(x_0+x_2)\otimes Y(x_0+x_2)\big)\big(1\otimes Y(x_2)\otimes 1\otimes Y(x_2)\big)R^{34}(-x_0-x_2)\\
&\qquad\cdot R^{23}(-x_0)R^{45}(-x_2)\big(u\otimes v\otimes u'\otimes v'\otimes u''\otimes v''\big).
\end{aligned}\tag{2.15}$$

On the other hand, using (2.11) we get

$$\begin{aligned}
&Y_R\big(Y_R(u\otimes v,x_0)\big(u'\otimes v'\big),x_2\big)\big(u''\otimes v''\big)\\
&\quad=\big(Y(x_2)\otimes Y(x_2)\big)R^{23}(-x_2)\big(Y(x_0)\otimes Y(x_0)\otimes 1\otimes 1\big)\\
&\qquad\cdot R^{23}(-x_0)\big(u\otimes v\otimes u'\otimes v'\otimes u''\otimes v''\big)\\
&\quad=\big(Y(x_2)\otimes Y(x_2)\big)\big(Y(x_0)\otimes 1\otimes Y(x_0)\otimes 1\big)R^{34}(-x_2-x_0)R^{45}(-x_2)\\
&\qquad\cdot R^{23}(-x_0)\big(u\otimes v\otimes u'\otimes v'\otimes u''\otimes v''\big)\\
&\quad=\big(Y(x_2)\otimes Y(x_2)\big)\big(Y(x_0)\otimes 1\otimes Y(x_0)\otimes 1\big)R^{34}(-x_2-x_0)R^{23}(-x_0)\\
&\qquad\cdot R^{45}(-x_2)\big(u\otimes v\otimes u'\otimes v'\otimes u''\otimes v''\big).
\end{aligned}\tag{2.16}$$

Then the desired weak associativity relation follows. Thus $(U\otimes V,Y_R,\mathbf{1}\otimes\mathbf{1})$ carries the structure of a nonlocal vertex algebra.

Furthermore, for $u,u'\in U$, as $R(x)(\mathbf{1}\otimes u')=u'\otimes\mathbf{1}$, we have

$$Y_R(u\otimes\mathbf{1},x)\big(u'\otimes\mathbf{1}\big)=Y(u,x)u'\otimes Y(\mathbf{1},x)\mathbf{1}=Y(u,x)u'\otimes\mathbf{1}.$$

It follows that the map $u\in U\mapsto u\otimes\mathbf{1}\in U\otimes V$ is a one-to-one homomorphism of nonlocal vertex algebras. Similarly, the map $v\in V\mapsto\mathbf{1}\otimes v\in U\otimes V$ is a one-to-one homomorphism of nonlocal vertex algebras. This concludes the proof. $\square$

Denote by $U\otimes_R V$ the nonlocal vertex algebra obtained in Theorem 2.4. We identify each element u of U with the element $u\otimes\mathbf{1}$ of $U\otimes_R V$ and identify each element v of V with $\mathbf{1}\otimes v$ of $U\otimes_R V$. For $u,u'\in U$, $v,v'\in V$, we have

$$Y_R(u,x)\big(u'\otimes v'\big)=Y(u,x)u'\otimes v',\tag{2.17}$$

$$Y_R(v,x)\big(u'\otimes v'\big)=\big(1\otimes Y(x)\big)R^{12}(-x)\big(v\otimes u'\otimes v'\big).\tag{2.18}$$

The following two propositions give more information about $U\otimes_R V$:

Proposition 2.5. *The $\mathcal{D}$-operator of $U\otimes_R V$ is given by*

$$\mathcal{D}_{U\otimes_R V}=\mathcal{D}\otimes 1+1\otimes\mathcal{D},\tag{2.19}$$

where the two $\mathcal{D}$'s denote the $\mathcal{D}$-operators of U and V, respectively. Furthermore, we have

$$Y_R(u,x)v \in (U \otimes V)[[x]], \tag{2.20}$$

$$Y_R(v,x)u = e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}Y_R(-x)R(-x)(v\otimes u) \tag{2.21}$$

for $u \in U$, $v \in V$.

Proof. Let $u \in U$, $v \in V$. From definition we have

$$\begin{aligned}\mathrm{Res}_x x^{-2}Y_R(u\otimes v,x)(\mathbf{1}\otimes\mathbf{1}) &= \mathrm{Res}_x x^{-2}\big(Y(u,x)\mathbf{1}\otimes Y(v,x)\mathbf{1}\big)\\ &= \mathrm{Res}_x x^{-2}e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}(u\otimes v)\\ &= (\mathcal{D}\otimes 1+1\otimes\mathcal{D})(u\otimes v).\end{aligned}$$

Thus the $\mathcal{D}$-operator is given by $\mathcal{D}\otimes 1+1\otimes\mathcal{D}$.

Let $u \in U$, $v \in V$. As

$$R^{23}(-x)(u\otimes\mathbf{1}\otimes\mathbf{1}\otimes v) = u\otimes\mathbf{1}\otimes\mathbf{1}\otimes v,$$

we have

$$Y_R(u,x)v = Y_R(u\otimes\mathbf{1},x)(\mathbf{1}\otimes v) = Y(u,x)\mathbf{1}\otimes Y(\mathbf{1},x)v = Y(u,x)\mathbf{1}\otimes v,$$

which implies (2.20) and

$$u\otimes v = u_{-1}\mathbf{1}\otimes v = (u\otimes\mathbf{1})_{-1}(\mathbf{1}\otimes v). \tag{2.22}$$

For $u \in U$, $v \in V$, write

$$R(x)(v\otimes u) = \sum_{i=1}^{r} u^{(i)}\otimes v^{(i)}\otimes f_i(x) \in U\otimes V\otimes\mathbb{C}((x))$$

(a finite sum). Using (2.5) we get

$$\begin{aligned}Y_R(\mathbf{1}\otimes v,x)(u\otimes\mathbf{1}) &= \big(Y(x)\otimes Y(x)\big)R^{23}(-x)(\mathbf{1}\otimes v\otimes u\otimes\mathbf{1})\\ &= \sum_{i=1}^{r} f_i(x)Y(\mathbf{1},x)u^{(i)}\otimes Y\big(v^{(i)},x\big)\mathbf{1}\\ &= \sum_{i=1}^{r} f_i(x)e^{x\mathcal{D}}Y\big(u^{(i)},-x\big)\mathbf{1}\otimes e^{x\mathcal{D}}Y(\mathbf{1},-x)v^{(i)}\\ &= e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}\big(Y(-x)\otimes Y(-x)\big)R^{14}(-x)(v\otimes\mathbf{1}\otimes\mathbf{1}\otimes u)\\ &= e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}\big(Y(-x)\otimes Y(-x)\big)R^{23}(x)R^{14}(-x)(v\otimes\mathbf{1}\otimes\mathbf{1}\otimes u)\\ &= e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}Y_R(-x)R^{14}(-x)(v\otimes\mathbf{1}\otimes\mathbf{1}\otimes u),\end{aligned}$$

as $R(-x)(\mathbf{1}\otimes\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}$ and $R^{23}(x)R^{14}(-x) = R^{14}(-x)R^{23}(x)$. This proves (2.21). □

Proposition 2.6. *Let U, V, $R(x)$ be given as in Theorem 2.4. For $u \in U$, $v \in V$, there exists a nonnegative integer k such that*

$$\begin{aligned}&(x_1-x_2)^k Y_R(v,x_1)Y_R(u,x_2)w\\&\quad=(x_1-x_2)^k Y_R(x_2)\big(1\otimes Y_R(x_1)\big)R^{12}(x_2-x_1)(v\otimes u\otimes w)\end{aligned}\tag{2.23}$$

for every $w \in U \otimes_R V$. Furthermore, if $R(x)$ is invertible, we also have

$$Y_R(u,x)v=e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}Y_R(-x)R^{-1}(x)(u\otimes v),\tag{2.24}$$

$$Y_R(u,x_1)Y_R(v,x_2)w=Y_R(x_2)\big(1\otimes Y_R(x_1)\big)\big(R^{-1}\big)^{12}(-x_2+x_1)(u\otimes v\otimes w)\tag{2.25}$$

for $u \in U$, $v \in V$ and for $w \in U \otimes_R V$.

Proof. For $u \in U$, $v \in V$, skew-symmetry relation (2.21) holds. From [L3] (Proposition 5.2), there exists a nonnegative integer k such that

$$\begin{aligned}&(x_1-x_2)^k Y_R(\mathbf{1}\otimes v,x_1)Y_R(u\otimes\mathbf{1},x_2)w\\&\quad=(x_1-x_2)^k Y_R(x_2)\big(1\otimes Y_R(x_1)\big)R^{14}(x_2-x_1)(v\otimes\mathbf{1}\otimes\mathbf{1}\otimes u\otimes w)\end{aligned}\tag{2.26}$$

for every $w \in U \otimes_R V$.

Now we assume that $R(x)$ is invertible. For $u \in U$, $v \in V$, we have

$$\begin{aligned}Y_R(u,x)v\ \big(&=Y_R(u\otimes\mathbf{1},x)(\mathbf{1}\otimes v)\big)\\&=\big(Y(x)\otimes Y(x)\big)R^{23}(-x)(u\otimes\mathbf{1}\otimes\mathbf{1}\otimes v)\\&=Y(u,x)\mathbf{1}\otimes Y(\mathbf{1},x)v\\&=e^{x\mathcal{D}}Y(\mathbf{1},-x)u\otimes e^{x\mathcal{D}}Y(v,-x)\mathbf{1}\\&=e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}\big(Y(-x)\otimes Y(-x)\big)(\mathbf{1}\otimes u\otimes v\otimes\mathbf{1})\\&=e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}Y_R(-x)\big(R^{-1}\big)^{23}(x)(\mathbf{1}\otimes u\otimes v\otimes\mathbf{1})\\&=e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}Y_R(-x)R^{-1}(x)(u\otimes v),\end{aligned}$$

proving (2.24). Again, from [L3] (Proposition 5.2), there exists a nonnegative integer k such that

$$(x_1-x_2)^k Y_R(u,x_1)Y_R(v,x_2)w=(x_1-x_2)^k Y_R(x_2)\big(1\otimes Y_R(x_1)\big)\big(R^{-1}\big)^{12}(-x_2+x_1)(u\otimes v\otimes w)$$

for every $w \in U \otimes_R V$. Combining this with weak associativity we get

$$\begin{aligned}&x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_R(u,x_1)Y_R(v,x_2)w\\&\quad-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_R(x_2)\big(1\otimes Y_R(x_1)\big)\big(R^{-1}\big)^{12}(-x_2+x_1)(u\otimes v\otimes w)\\&=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_R\big(Y_R(u,x_0)v,x_2\big)w.\end{aligned}$$

Note that $Y_R(u,x_0)v \in (U \otimes V)[[x_0]]$ by (2.20). Applying Res_{x_0} to the above Jacobi identity we obtain (2.25). $\square$

The twisted tensor product $U \otimes_R V$ has a universal property just as the ordinary tensor product $U \otimes V$ does (cf. [FHL,LL]).

Theorem 2.7. *Let U, V be nonlocal vertex algebras and let $R(x)$ be a twisting operator for (U,V). Let K be any nonlocal vertex algebra and let $\psi_1 : U \to K$, $\psi_2 : V \to K$ be any homomorphisms, satisfying the condition that for $u \in U$, $v \in V$,*

$$Y\big(\psi_1(u),x\big)\psi_2(v) \in K[[x]], \tag{2.27}$$

$$Y\big(\psi_2(v),x\big)\psi_1(u) = e^{x\mathcal{D}}Y(-x)(\psi_1 \otimes \psi_2)R(-x)(v \otimes u). \tag{2.28}$$

Then the linear map $\psi : U \otimes_R V \to K$, defined by

$$\psi(u \otimes v) = \psi_1(u)_{-1}\psi_2(v) \quad \textit{for } u \in U,\ v \in V,$$

is a homomorphism of nonlocal vertex algebras, which extends ψ_1 and ψ_2 uniquely.

Proof. For $u \in U$, $v \in V$, as $Y(\psi_1(u),x)\psi_2(v) \in K[[x]]$ by assumption, we have

$$\psi(u \otimes v) = \mathrm{Res}_x x^{-1} Y\big(\psi_1(u),x\big)\psi_2(v) = \lim_{x \to 0} Y\big(\psi_1(u),x\big)\psi_2(v).$$

It is clear that linear map ψ extends both ψ_1 and ψ_2. It is also clear that $\psi(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1}$. To prove that ψ is a homomorphism of nonlocal vertex algebras, we must prove

$$\psi\big(Y_R(u \otimes v,x)\big(u' \otimes v'\big)\big) = Y\big(\psi(u \otimes v),x\big)\psi\big(u' \otimes v'\big) \quad \text{for } u,u' \in U,\ v,v' \in V. \tag{2.29}$$

Through the homomorphisms ψ_1 and ψ_2, K becomes a U-module and a V-module. Let $v \in V$. By assumption, we have $Y(\psi_1(u),x)\psi_2(v) \in K[[x]]$ for every $u \in U$. By a result of [L3] (Lemma 6.1), for every fixed $v \in V$, the map $U \ni u \mapsto \psi_1(u)_{-1}\psi_2(v) \in K$ is a U-module homomorphism. Then, for $u',u \in U$, $v \in V$, we have

$$\begin{aligned}\psi\big(Y_R\big(u',x_0\big)(u \otimes v)\big) &= \psi\big(Y\big(u',x_0\big)u \otimes v\big) = \lim_{x \to 0} Y\big(\psi_1 Y\big(u',x_0\big)u,x\big)\psi_2(v)\\ &= Y\big(u',x_0\big)\big(\psi_1(u)_{-1}\psi_2(v)\big) = Y\big(\psi_1\big(u'\big),x_0\big)\psi(u \otimes v)\\ &= Y\big(\psi\big(u' \otimes \mathbf{1}\big),x_0\big)\psi(u \otimes v).\end{aligned}$$

This shows that (2.29) holds with $v = \mathbf{1}$. We next show that (2.29) also holds with $u = \mathbf{1}$. We have

$$\begin{aligned}\psi\big(Y_R(\mathbf{1} \otimes v,x)\big(u' \otimes v'\big)\big) &= \psi\big(\big(1 \otimes Y(x)\big)R^{12}(-x)\big(v \otimes u' \otimes v'\big)\big)\\ &= \lim_{x_2 \to 0} Y(x_2)\big(1 \otimes Y(x)\big)(\psi_1 \otimes \psi_2 \otimes \psi_2)R^{12}(-x)\big(v \otimes u' \otimes v'\big).\end{aligned} \tag{2.30}$$

With the assumption (2.28), from [L3] (Proposition 5.2), there exists a nonnegative integer k such that

$$\begin{aligned}&(x_2 - x)^k Y\big(\psi_2(v),x\big)Y\big(\psi_1\big(u'\big),x_2\big)\psi_2\big(v'\big)\\ &\quad = (x_2 - x)^k Y(x_2)\big(1 \otimes Y(x)\big)(\psi_1 \otimes \psi_2 \otimes \psi_2)R^{12}(x_2 - x)\big(v \otimes u' \otimes v'\big).\end{aligned}$$

Noticing that $R(x)(v\otimes u')\in U\otimes V\otimes\mathbb{C}((x))$, we may replace k with a bigger integer so that $x^kR(x)(v\otimes u')\in U\otimes V\otimes\mathbb{C}[[x]]$ also holds. Then

$$\begin{aligned}
&(-x)^kY\big(\psi(\mathbf{1}\otimes v),x\big)\psi\big(u'\otimes v'\big)\\
&\quad=\lim_{x_2\to 0}(x_2-x)^kY\big(\psi_2(v),x\big)Y\big(\psi_1\big(u'\big),x_2\big)\psi_2\big(v'\big)\\
&\quad=\lim_{x_2\to 0}(x_2-x)^kY(x_2)\big(1\otimes Y(x)\big)(\psi_1\otimes\psi_2\otimes\psi_2)R^{12}(x_2-x)\big(v\otimes u'\otimes v'\big)\\
&\quad=(-x)^k\lim_{x_2\to 0}Y(x_2)\big(1\otimes Y(x)\big)(\psi_1\otimes\psi_2\otimes\psi_2)R^{12}(-x)\big(v\otimes u'\otimes v'\big).
\end{aligned}$$

Thus

$$Y\big(\psi(\mathbf{1}\otimes v),x\big)\psi\big(u'\otimes v'\big)=\lim_{x_2\to 0}Y(x_2)\big(1\otimes Y(x)\big)(\psi_1\otimes\psi_2\otimes\psi_2)R^{12}(-x)\big(v\otimes u'\otimes v'\big).$$

Combining this with (2.30) we obtain

$$\psi\big(Y_R(\mathbf{1}\otimes v,x)\big(u'\otimes v'\big)\big)=Y\big(\psi(\mathbf{1}\otimes v),x\big)\psi\big(u'\otimes v'\big),$$

proving that (2.29) holds with $u=\mathbf{1}$. Since $U\otimes_R V$ as a nonlocal vertex algebra is generated by the subset $U\cup V$, it follows that ψ is a homomorphism of nonlocal vertex algebras. The uniqueness assertion is clear as $u\otimes v=(u\otimes\mathbf{1})_{-1}(\mathbf{1}\otimes v)$ for $u\in U$, $v\in V$. □

The following is a characterization of $U\otimes_R V$ in terms of U, V and $R(x)$:

Proposition 2.8. *Let U, V and $R(x)$ be given as in Theorem* 2.7, *and let K be a nonlocal vertex algebra which contains U and V as subalgebras, satisfying*

$$\begin{aligned}
&Y(u,x)v\in K[[x]],\\
&Y(v,x)u=e^{x\mathcal{D}}Y(-x)R(-x)(v\otimes u)\quad\text{for } u\in U,\ v\in V.
\end{aligned}$$

Assume that K as a nonlocal vertex algebra is generated by $U\cup V$ and that U as a U-module is irreducible and of countable dimension (*over* $\mathbb{C}$)*. Then the linear map $\theta: U\otimes_R V\to K$, defined by $\theta(u\otimes v)=u_{-1}v$ for $u\in U$, $v\in V$, is a nonlocal-vertex-algebras isomorphism.*

Proof. It follows from Theorem 2.7 that θ is a nonlocal-vertex-algebra homomorphism. Now we prove that θ is a bijection. Since $U\otimes_R V$ as a nonlocal vertex algebra is generated by $U\cup V$ and since K is also generated by $U\cup V$, it follows that θ is onto. On the other hand, as $\theta|_U=1$, θ is a U-module homomorphism. Consequently, $\ker\theta$ is a U-submodule of $U\otimes_R V$. From (2.17), for any subspace P of V, $U\otimes P$ is a U-submodule. As U as a U-module is irreducible and of countable dimension (over $\mathbb{C}$), by a version of Schur lemma (cf. [CG]) we have $\mathrm{End}_U U=\mathbb{C}$. It follows that $\ker\theta=U\otimes B$ for some subspace B of V. For any $b\in B$, we have $\mathbf{1}\otimes b\in\ker\theta$, so that $b=\theta(\mathbf{1}\otimes b)=0$. Thus $B=0$ and $\ker\theta=0$, proving that θ is injective. □

Recall from Lemma 2.3 that for any invertible twisting operator $R(x)$ for (U,V), $R^{-1}(-x)$ is an invertible twisting operator for (V,U). Furthermore, we have:

Proposition 2.9. *Let $R(x)$ be an invertible twisting operator for (U, V) such that*

$$R(x)(v \otimes u) \in U \otimes V \otimes \mathbb{C}[[x]], \qquad R^{-1}(x)(u \otimes v) \in V \otimes U \otimes \mathbb{C}[[x]] \tag{2.31}$$

for $u \in U$, $v \in V$. Then the linear map $\psi : V \otimes_{R^{-1}(-x)} U \to U \otimes_R V$, defined by

$$\psi(v \otimes u) = v_{-1}u \quad (\text{in } U \otimes_R V) \quad \text{for } v \in V,\ u \in U,$$

is a nonlocal-vertex-algebra isomorphism.

Proof. With the assumption (2.31), combining (2.21) with (2.20) we get

$$Y_R(v, x)u \in (U \otimes_R V)[[x]] \quad \text{for } u \in U,\ v \in V. \tag{2.32}$$

From Proposition 2.6 we also have

$$Y_R(u, x)v = e^{x(\mathcal{D}\otimes 1 + 1\otimes \mathcal{D})} Y_R(-x) R^{-1}(x)(u \otimes v).$$

By Theorem 2.7, ψ is a nonlocal-vertex-algebra homomorphism. Clearly, $\psi|_U = 1$ and $\psi|_V = 1$. On the other hand, consider $V \otimes_{R^{-1}(-x)} U$ and denote $Y_{R^{-1}(-x)}$ simply by $Y_{R^{-1}}$. For $u \in U$, $v \in V$, by (2.20) and (2.21) we have

$$Y_{R^{-1}}(v, x)u \in (V \otimes U)[[x]],$$
$$Y_{R^{-1}}(u, x)v = e^{x(\mathcal{D}\otimes 1 + 1\otimes \mathcal{D})} Y_{R^{-1}}(-x) R^{-1}(x)(u \otimes v).$$

Combining these with (2.31) we get

$$Y_{R^{-1}}(u, x)v \in (V \otimes U)[[x]],$$

while from Proposition 2.6 we have

$$Y_{R^{-1}}(v, x)u = e^{x(\mathcal{D}\otimes 1 + 1\otimes \mathcal{D})} Y_{R^{-1}}(-x) R(-x)(v \otimes u).$$

By Theorem 2.7, there is a nonlocal-vertex-algebra homomorphism $\phi : U \otimes_R V \to V \otimes_{R^{-1}(-x)} U$ such that $\phi(u \otimes v) = u_{-1}v$ for $u \in U$, $v \in V$. Because $\psi \circ \phi$ and $\phi \circ \psi$ are nonlocal-vertex-algebra homomorphisms preserving both U and V element-wise, it follows that $\psi \circ \phi = 1$ and $\phi \circ \psi = 1$. Therefore, ψ is a nonlocal-vertex-algebra isomorphism. $\square$

Next, we shall establish a refinement of Proposition 2.8. As we need, we recall an important notion due to Etingof and Kazhdan. A nonlocal vertex algebra V is said to be *non-degenerate* (see [EK]) if for every positive integer n, the linear map

$$Z_n : V^{\otimes n} \otimes \mathbb{C}((x_1)) \cdots ((x_n)) \to V((x_1)) \cdots ((x_n)),$$

defined by

$$Z_n\big(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f\big) = f Y\big(v^{(1)}, x_1\big) \cdots Y\big(v^{(n)}, x_n\big)\mathbf{1}$$

for $v^{(1)}, \ldots, v^{(n)} \in V$, $f \in \mathbb{C}((x_1)) \cdots ((x_n))$, is injective.

Following [L3], for every positive integer n, we define a linear map

$$\pi_n : V^{\otimes n} \otimes \mathbb{C}((x_1)) \cdots ((x_n)) \to \mathrm{Hom}\big(V, V((x_1)) \cdots ((x_n))\big)$$

by

$$\pi_n\big(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f\big)(w) = f Y\big(v^{(1)}, x_1\big) \cdots Y\big(v^{(n)}, x_n\big) w$$

for $v^{(1)}, \ldots, v^{(n)} \in V$, $f \in \mathbb{C}((x_1)) \cdots ((x_n))$ and for $w \in V$. By definition we have

$$Z_n\big(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f\big) = \pi_n\big(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f\big)(\mathbf{1}).$$

We see that π_n is injective if Z_n is injective. In particular, non-degeneracy implies that all the linear maps π_n for $n \geqslant 1$ are injective.

For convenience we recall a notion from [L3]. Let P be a vector space and let r be a positive integer. For $A, B \in P[[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_r^{\pm 1}]]$, we define $A \sim B$ if there exists a nonzero polynomial $p(x_1, \ldots, x_r)$ such that

$$p(x_1, \ldots, x_r) A(x_1, \ldots, x_r) = p(x_1, \ldots, x_r) B(x_1, \ldots, x_r).$$

This is an equivalence relation. Furthermore, when restricted onto the subspace $P((x_r)) \cdots ((x_2))((x_1))$, this equivalence relation becomes equality relation.

Just as with bialgebras, a braided tensor product nonlocal vertex algebra arises whenever a nonlocal vertex algebra contains two compatible subalgebras. The following can be considered as an analogue of a result in [M]:

Theorem 2.10. *Let K be a nonlocal vertex algebra that contains subalgebras U and V, satisfying the condition that for $u \in U$, $v \in V$,*

$$Y(u, x)v \in K[[x]] \tag{2.33}$$

and there exist

$$u^{(i)} \in U, \qquad v^{(i)} \in V, \qquad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \ldots, r)$$

such that

$$(x_1 - x_2)^k Y(v, x_1) Y(u, x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} f_i(x_2 - x_1) Y\big(u^{(i)}, x_2\big) Y\big(v^{(i)}, x_1\big) \tag{2.34}$$

for some nonnegative integer k. Assume that K is non-degenerate and that K is generated by $U \cup V$. Then there exists a linear map $R(x) : V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$, which is uniquely determined by the condition that

$$Y(v, x_1) Y(u, x_2) w \sim Y(x_2)\big(1 \otimes Y(x_1)\big) R^{12}(x_2 - x_1)(v \otimes u \otimes w) \tag{2.35}$$

for $u \in U$, $v \in V$, $w \in K$, and $R(x)$ is a twisting operator. Furthermore, the linear map $\psi : U \otimes V \to K$, defined by

$$\psi(u \otimes v) = u_{-1} v \quad \text{for } u \in U,\ v \in V,$$

is a nonlocal-vertex-algebra isomorphism from $U \otimes_R V$ onto K.

Proof. From the assumption, there exists a linear map $R(x): V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$, satisfying (2.35). Suppose that $T(x)$ is another such linear map. For $u \in U$, $v \in V$, $w \in K$, we have

$$Y(x_2)\big(1 \otimes Y(x_1)\big)R^{12}(x_2 - x_1)(v \otimes u \otimes w) \sim Y(x_2)\big(1 \otimes Y(x_1)\big)T^{12}(x_2 - x_1)(v \otimes u \otimes w).$$

As the expressions on both sides lie in $(U \otimes V)((x_2))((x_1))$, we must have

$$Y(x_2)\big(1 \otimes Y(x_1)\big)R^{12}(x_2 - x_1)(v \otimes u \otimes w) = Y(x_2)\big(1 \otimes Y(x_1)\big)T^{12}(x_2 - x_1)(v \otimes u \otimes w).$$

Given that K is non-degenerate, we know π_2 is injective. It follows that $R(x)(v \otimes u) = T(x)(v \otimes u)$. This proves the uniqueness.

For $u \in U$, by definition we have

$$Y(\mathbf{1}, x_1)Y(u, x_2)w \sim Y(x_2)\big(1 \otimes Y(x_1)\big)R^{12}(x_2 - x_1)(\mathbf{1} \otimes u \otimes w).$$

On the other hand, as $Y(\mathbf{1}, x) = 1$ we have

$$Y(\mathbf{1}, x_1)Y(u, x_2)w = Y(u, x_2)Y(\mathbf{1}, x_1)w = Y(x_2)\big(1 \otimes Y(x_1)\big)(u \otimes \mathbf{1} \otimes w).$$

Thus

$$Y(x_2)\big(1 \otimes Y(x_1)\big)R^{12}(x_2 - x_1)(\mathbf{1} \otimes u \otimes w) \sim Y(x_2)\big(1 \otimes Y(x_1)\big)(u \otimes \mathbf{1} \otimes w).$$

As we have seen above, this equivalence relation implies equality relation

$$Y(x_2)\big(1 \otimes Y(x_1)\big)R^{12}(x_2 - x_1)(\mathbf{1} \otimes u \otimes w) = Y(x_2)\big(1 \otimes Y(x_1)\big)(u \otimes \mathbf{1} \otimes w).$$

Just as above, with π_2 being injective it follows that $R(\mathbf{1} \otimes u) = u \otimes \mathbf{1}$. Using a parallel argument we get that $R(x)(v \otimes \mathbf{1}) = \mathbf{1} \otimes v$ for $v \in V$.

Now, let $v \in V$, $u, u' \in U$ and let $w \in K$. We have

$$\begin{aligned}
&Y(v, z_1)Y(u, x_1)Y\big(u', x_2\big)w \\
&\quad \sim Y(x_1)\big(1 \otimes Y(z_1)\big)\big(1 \otimes 1 \otimes Y(x_2)\big)R^{12}(x_1 - z_1)\big(v \otimes u \otimes u' \otimes w\big) \\
&\quad \sim Y(x_1)\big(1 \otimes Y(x_2)\big)\big(1 \otimes 1 \otimes Y(z_1)\big)R^{23}(x_2 - z_1)R^{12}(x_1 - z_1)\big(v \otimes u \otimes u' \otimes w\big).
\end{aligned}$$

Furthermore, by Lemma 2.1 there exists a nonnegative integer k such that

$$\begin{aligned}
&\big((x_1 - x_2)^k Y(x_1)\big(1 \otimes Y(x_2)\big)\big(1 \otimes 1 \otimes Y(z_1)\big) \\
&\qquad \cdot R^{23}(x_2 - z_1)R^{12}(x_1 - z_1)\big(v \otimes u \otimes u' \otimes w\big)\big)\big|_{x_1 = x_2 + x_0} \\
&\quad = x_0^k Y(x_2)\big(Y(x_0) \otimes 1\big)\big(1 \otimes 1 \otimes Y(z_1)\big)R^{23}(x_2 - z_1)R^{12}(x_2 + x_0 - z_1)\big(v \otimes u \otimes u' \otimes w\big).
\end{aligned}$$

Thus

$$\begin{aligned}
&\big((x_1 - x_2)^k Y(v, z_1)Y(u, x_1)Y\big(u', x_2\big)w\big)\big|_{x_1 = x_2 + x_0} \\
&\quad \sim x_0^k Y(x_2)\big(Y(x_0) \otimes 1\big)\big(1 \otimes 1 \otimes Y(z_1)\big)R^{23}(x_2 - z_1)R^{12}(x_2 + x_0 - z_1)\big(v \otimes u \otimes u' \otimes w\big) \\
&\quad = x_0^k Y(x_2)\big(1 \otimes Y(z_1)\big)\big(Y(x_0) \otimes 1 \otimes 1\big)R^{23}(x_2 - z_1)R^{12}(x_2 - z_1 + x_0)\big(v \otimes u \otimes u' \otimes w\big). \quad (2.36)
\end{aligned}$$

On the other hand, by Lemma 2.1 there exists a nonnegative integer k' such that

$$(x_1 - x_2)^{k'} Y(u, x_1) Y(u', x_2) \in \mathrm{Hom}\big(K, K((x_1, x_2))\big)$$

and

$$\big((x_1 - x_2)^{k'} Y(u, x_1) Y(u', x_2) w\big)\big|_{x_1 = x_2 + x_0} = x_0^{k'} Y\big(Y(u, x_0)u', x_2\big) w.$$

Then

$$\begin{aligned}
&\big((x_1 - x_2)^{k'} Y(v, z_1) Y(u, x_1) Y(u', x_2) w\big)\big|_{x_1 = x_2 + x_0} \\
&\quad = x_0^{k'} Y(v, z_1) Y\big(Y(u, x_0)u', x_2\big) w \\
&\quad = x_0^{k'} Y(z_1)\big(1 \otimes Y(x_2)\big)\big(1 \otimes Y(x_0) \otimes 1\big)\big(v \otimes u \otimes u' \otimes w\big) \\
&\quad \sim x_0^{k'} Y(x_2)\big(1 \otimes Y(z_1)\big) R^{12}(x_2 - z_1)\big(1 \otimes Y(x_0) \otimes 1\big)\big(v \otimes u \otimes u' \otimes w\big).
\end{aligned} \tag{2.37}$$

Combining this with (2.36) we get

$$\begin{aligned}
&Y(x_2)\big(1 \otimes Y(z_1)\big) R^{12}(x_2 - z_1)\big(1 \otimes Y(x_0) \otimes 1\big)\big(v \otimes u \otimes u' \otimes w\big) \\
&\quad \sim Y(x_2)\big(1 \otimes Y(z_1)\big)\big(Y(x_0) \otimes 1 \otimes 1\big) R^{23}(x_2 - z_1) R^{12}(x_2 - z_1 + x_0)\big(v \otimes u \otimes u' \otimes w\big).
\end{aligned}$$

Because both sides lie in $K((x_2))((z_1))((x_0))$, this similarity relation implies equality relation. Then, with π_2 injective we have

$$R(x_2 - z_1)\big(1 \otimes Y(x_0)\big)\big(v \otimes u \otimes u'\big) = \big(Y(x_0) \otimes 1\big) R^{23}(x_2 - z_1) R^{12}(x_2 - z_1 + x_0)\big(v \otimes u \otimes u'\big).$$

This proves

$$R(x)\big(1 \otimes Y(x_0)\big) = \big(Y(x_0) \otimes 1\big) R^{23}(x) R^{12}(x + x_0),$$

confirming (2.10). The other condition (2.11) can be proved in the same manner. Therefore, R is a twisting operator for the ordered pair (U, V).

As for the last assertion, it follows from Theorem 2.7 that ψ is a nonlocal-vertex-algebra homomorphism, and we have $\psi|_U = 1$ and $\psi|_V = 1$. Since K is generated by $U \cup V$, it follows that ψ is onto. For $u \in U$, $v \in V$, we have $Y(u, x)v \in K[[x]]$ and $[\mathcal{D}, Y(u, x)] = \frac{d}{dx} Y(u, x)$, which imply $Y(u, x)v = e^{x\mathcal{D}} u_{-1} v$. From this we get $\ker \psi \subset \ker Y(x)$. Now we show that $\ker Y(x) = 0$. For $a, b \in K$, by weak associativity, we have

$$(x_0 + x_2)^l Y(a, x_0 + x_2) Y(b, x_2)\mathbf{1} = (x_0 + x_2)^l Y\big(Y(a, x_0)b, x_2\big)\mathbf{1}$$

for some nonnegative integer l. As

$$Y(a, x_0 + x_2) Y(b, x_2)\mathbf{1},\ \ Y\big(Y(a, x_0)b, x_2\big)\mathbf{1} \in K((x_0))[[x_2]],$$

we must have

$$Y(a, x_0 + x_2) Y(b, x_2)\mathbf{1} = Y\big(Y(a, x_0)b, x_2\big)\mathbf{1}.$$

From this we have $\ker Y(x) \subset \ker Z_2$. With K non-degenerate, we have $\ker Z_2 = 0$, so that $\ker Y(x) = 0$ and $\ker \psi = 0$. Thus ψ is also injective. Therefore, ψ is an isomorphism. □

Note that as U and V are subalgebras of $U \otimes_R V$, every $U \otimes_R V$-module is naturally a U-module and a V-module. Furthermore, we have:

Lemma 2.11. *Let (W, Y_W) be a $U \otimes_R V$-module and let $u \in U$, $v \in V$. Then*

$$Y_W(u, x_1)Y_W(v, x_2) \in \operatorname{Hom}\bigl(W, W((x_1, x_2))\bigr), \tag{2.38}$$

and there exists a nonnegative integer k such that

$$\begin{aligned}&(x_2 - x_1)^k Y_W(v, x_1)Y_W(u, x_2)w\\&\quad = (x_2 - x_1)^k Y_W(x_2)\bigl(1 \otimes Y_W(x_1)\bigr)R^{12}(x_2 - x_1)(v \otimes u \otimes w)\end{aligned} \tag{2.39}$$

for every $w \in W$. If R is invertible, we also have

$$Y_W(u, x_1)Y_W(v, x_2)w = Y_W(x_2)\bigl(1 \otimes Y_W(x_1)\bigr)\bigl(R^{-1}\bigr)^{12}(x_2 - x_1)(u \otimes v \otimes w). \tag{2.40}$$

Proof. Let $u \in U$, $v \in V$ and let $w \in W$. There exists $l \in \mathbb{N}$ such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w = (x_0 + x_2)^l Y_W\bigl(Y_R(u, x_0)v, x_2\bigr)w.$$

As $Y_R(u, x_0)v \in (U \otimes_R V)[[x_0]]$, the expression on the right side lies in $W((x_2))[[x_0]]$. This forces the expression on the left side to lie in $W((x_2))[[x_0]] \cap W((x_0))((x_2))$. Thus

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w \in W[[x_0, x_2]]\bigl[x_2^{-1}\bigr].$$

Applying $e^{-x_2\partial/\partial x_0}$ we get

$$x_0^l Y_W(u, x_0)Y_W(v, x_2)w \in W[[x_0, x_2]]\bigl[x_2^{-1}\bigr],$$

which implies $Y_W(u, x_0)Y_W(v, x_2)w \in W((x_0, x_2))$. This proves (2.38).

For $u \in U$, $v \in V$, skew symmetry relation (2.21) holds. Then the second assertion follows immediately from [L3] (Proposition 5.2).

Assume that R is invertible. Then (2.25) (in Proposition 2.6) holds. By Corollary 5.4 of [L3], there exists a nonnegative integer k such that

$$(x_2 - x_1)^k Y_W(u, x_1)Y_W(v, x_2)w = (x_2 - x_1)^k Y_W(x_2)\bigl(1 \otimes Y_W(x_1)\bigr)\bigl(R^{-1}\bigr)^{12}(x_2 - x_1)(u \otimes v \otimes w),$$

which together with weak associativity gives

$$\begin{aligned}&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_W(u, x_1)Y_W(v, x_2)w\\&\quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y_W(x_2)\bigl(1 \otimes Y_W(x_1)\bigr)\bigl(R^{-1}\bigr)^{12}(x_2 - x_1)(u \otimes v \otimes w)\\&= x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)Y_W\bigl(Y_R(u, x_0)v, x_2\bigr)w.\end{aligned}$$

Since $Y_R(u, x_0)v \in (U \otimes_R V)[[x_0]]$, applying Res_{x_0} we obtain (2.40). □

On the other hand, we show that a U-module structure together with a compatible V-module structure on the same space gives rise to a $U \otimes_R V$-module structure.

Proposition 2.12. *Let U and V be nonlocal vertex algebras and let $R(x)$ be an invertible twisting operator for (U, V). Let W be a vector space equipped with a U-module structure (W, Y_W^U) and a V-module structure (W, Y_W^V). Assume that for $u \in U$, $v \in V$,*

$$Y_W^U(u, x_1)Y_W^V(v, x_2) \in \operatorname{Hom}\big(W, W((x_1, x_2))\big), \tag{2.41}$$

(2.40) *holds, and there exists a nonnegative integer k such that*

$$\begin{aligned}(x_2 - x_1)^k Y_W^V(v, x_1)Y_W^U(u, x_2)w \\ = (x_2 - x_1)^k Y_W^U(x_2)\big(1 \otimes Y_W^V(x_1)\big)R^{12}(x_2 - x_1)(v \otimes u \otimes w)\end{aligned} \tag{2.42}$$

for every $w \in W$. Then there exists a module structure Y_R^W on W for $U \otimes_R V$, extending Y_W^U and Y_W^V uniquely.

Proof. For $u \in U$, $v \in V$, we define $Y_R^W(u \otimes v, x) \in (\operatorname{End} W)[[x, x^{-1}]]$ by

$$Y_R^W(u \otimes v, x)w = \big(Y_W^U(u, x_1)Y_W^V(v, x)w\big)\big|_{x_1 = x}$$

for $w \in W$. Notice that since $Y_W^U(u, x_1)Y_W^V(v, x)w \in W((x_1, x))$ by assumption,

$$\big(Y_W^U(u, x_1)Y_W^V(v, x)w\big)\big|_{x_1 = x}$$

exists in $W((x))$. Thus $Y_R^W(u \otimes v, x)$ is well defined as an element of $\operatorname{Hom}(W, W((x)))$. It is clear that $Y_R^W(\mathbf{1} \otimes \mathbf{1}, x) = 1_W$.

To establish weak associativity, let $u, u' \in U$, $v, v' \in V$ and let $w \in W$. We shall use Lemma 2.1. By definition we have

$$\begin{aligned}Y_R^W(u \otimes v, x_1)Y_R^W\big(u' \otimes v', x_2\big)w \\ = \big(Y_W^U(u, z_1)Y_W^V(v, x_1)Y_W^U\big(u', z_2\big)Y_W^V\big(v', x_2\big)w\big)\big|_{z_1 = x_1,\, z_2 = x_2}.\end{aligned} \tag{2.43}$$

We next show that there exists a nonnegative integer k independent of w such that

$$p(z_1, x_1, z_2, x_2)^k Y_W^U(u, z_1)Y_W^V(v, x_1)Y_W^U\big(u', z_2\big)Y_W^V\big(v', x_2\big)w \in W((z_1, x_1, z_2, x_2)), \tag{2.44}$$

where

$$p(z_1, x_1, z_2, x_2) = (z_1 - z_2)(x_1 - z_2)(x_1 - x_2).$$

From assumption we have

$$Y_W^U(u, z_1)Y_W^V(v, x_1)Y_W^U\big(u', z_2\big)Y_W^V\big(v', x_2\big)w \in W((z_1, x_1))((z_2, x_2)). \tag{2.45}$$

Let k be a nonnegative integer independent of w such that (2.42) with u' in place of u holds and such that $x^k R(x)(v \otimes u') \in U \otimes V \otimes \mathbb{C}[[x]]$. In view of Lemma 2.1 for (W, Y_W^V), we may also assume that

$$(x_1-x_2)^k\big(1\otimes 1\otimes Y_W^V(x_1)\big)\big(1\otimes 1\otimes 1\otimes Y_W^V(x_2)\big)R^{23}(\xi)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)$$
$$\in U\otimes U\otimes W((x_1,x_2))\otimes \mathbb{C}((\xi)).$$

Then

$$\begin{aligned}
&(x_1-x_2)^k(z_2-x_1)^k Y_W^U(u,z_1)Y_W^V(v,x_1)Y_W^U\big(u',z_2\big)Y_W^V\big(v',x_2\big)w\\
&\quad=(x_1-x_2)^k Y_W^U(z_1)\big(1\otimes Y_W^U(z_2)\big)\big(1\otimes 1\otimes Y_W^V(x_1)\big)\big(1\otimes 1\otimes 1\otimes Y_W^V(x_2)\big)\\
&\qquad\cdot (z_2-x_1)^k R^{23}(z_2-x_1)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)\\
&\quad\in W((z_1))((z_2,x_1,x_2))
\end{aligned}\tag{2.46}$$

(recall (2.45)). From assumption (2.40) we also have

$$\begin{aligned}
&Y_W^U(u,z_1)Y_W^V(v,x_1)Y_W^U\big(u',z_2\big)Y_W^V\big(v',x_2\big)w\\
&\quad=Y_W^V(x_1)\big(1\otimes Y_W^U(z_1)\big)\big(1\otimes 1\otimes Y_W^U(z_2)\big)\big(1\otimes 1\otimes 1\otimes Y_W^V(x_2)\big)\\
&\qquad\cdot\big(R^{-1}\big)^{12}(x_1-z_1)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)\\
&\quad=Y_W^V(x_1)\big(1\otimes Y_W^U(z_1)\big)\big(1\otimes 1\otimes Y_W^V(x_2)\big)\big(1\otimes 1\otimes 1\otimes Y_W^U(z_2)\big)\\
&\qquad\cdot\big(R^{-1}\big)^{34}(x_2-z_2)\big(R^{-1}\big)^{12}(x_1-z_1)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)\\
&\quad=Y_W^V(x_1)\big(1\otimes Y_W^V(x_2)\big)\big(1\otimes 1\otimes Y_W^U(z_1)\big)\big(1\otimes 1\otimes 1\otimes Y_W^U(z_2)\big)\\
&\qquad\cdot\big(R^{-1}\big)^{23}(x_2-z_1)\big(R^{-1}\big)^{34}(x_2-z_2)\big(R^{-1}\big)^{12}(x_1-z_1)\big(u\otimes v\otimes u'\otimes v'\otimes w\big).
\end{aligned}\tag{2.47}$$

We may choose k (independent of w) so large that

$$\begin{aligned}
&(z_1-z_2)^k\big(1\otimes 1\otimes Y_W^U(z_1)\big)\big(1\otimes 1\otimes 1\otimes Y_W^U(z_2)\big)\\
&\qquad\cdot\big(R^{-1}\big)^{23}(\xi_1)\big(R^{-1}\big)^{34}(\xi_2)\big(R^{-1}\big)^{12}(\xi_3)\big(u\otimes v\otimes u'\otimes v'\big)\\
&\quad\in V\otimes V\otimes \mathbb{C}((\xi_1,\xi_2,\xi_3))\otimes W((z_1,z_2)).
\end{aligned}$$

Then from (2.47) we see that

$$(z_1-z_2)^k Y_W^U(u,z_1)Y_W^V(v,x_1)Y_W^U\big(u',z_2\big)Y_W^V\big(v',x_2\big)w$$

lies in $W((x_1))((x_2,z_1,z_2))$. Combining this with (2.46) we obtain (2.44).

Using (2.43) and (2.44) we get

$$\begin{aligned}
&(x_1-x_2)^{3k}Y_R^W(u\otimes v,x_1)Y_R^W\big(u'\otimes v',x_2\big)w\\
&\quad=\big(p(z_1,x_1,z_2,x_2)^k Y_W^U(u,z_1)Y_W^V(v,x_1)Y_W^U\big(u',z_2\big)Y_W^V\big(v',x_2\big)w\big)\big|_{z_1=x_1,\,z_2=x_2}\\
&\quad\in W((x_1,x_2)),
\end{aligned}$$

and we have

$$\begin{aligned}
&\big(p(z_1,x_1,z_2,x_2)^k Y_W^U(u,z_1)Y_W^V(v,x_1)Y_W^U(u',z_2)Y_W^V(v',x_2)w\big)\big|_{z_1=z_2+x_0,\,x_1=x_2+x_0}\\
&\quad=(z_1-z_2)^k(x_1-x_2)^k\big(Y_W^U(z_1)\big(1\otimes Y_W^U(z_2)\big)\big(1\otimes 1\otimes Y_W^V(x_1)\big)\big(1\otimes 1\otimes 1\otimes Y_W^V(x_2)\big)\\
&\qquad\cdot(z_2-x_1)^k R^{23}(z_2-x_1)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)\big)\big|_{z_1=z_2+x_0,\,x_1=x_2+x_0}\\
&\quad=(z_1-z_2)^k(x_1-x_2)^k Y_W^U(z_1)\big(1\otimes Y_W^U(z_2)\big)\big(1\otimes 1\otimes Y_W^V(x_1)\big)\big(1\otimes 1\otimes 1\otimes Y_W^V(x_2)\big)\\
&\qquad\cdot(x_1-z_2)^k R^{23}(-x_1+z_2)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)\big|_{z_1=z_2+x_0,\,x_1=x_2+x_0}\\
&\quad=x_0^{2k}Y_W^U(z_2)\big(1\otimes Y_W^V(x_2)\big)\big(Y(x_0)\otimes Y(x_0)\otimes 1\big)\\
&\qquad\cdot(x_0-z_2+x_2)^k R^{23}(-x_0+z_2-x_2)\big(u\otimes v\otimes u'\otimes v'\otimes w\big).
\end{aligned}$$

Using substitution $z_2=x_2$ we get

$$\begin{aligned}
&\big((x_1-x_2)^{3k}Y_R^W(u\otimes v,x_1)Y_R^W\big(u'\otimes v',x_2\big)w\big)\big|_{x_1=x_2+x_0}\\
&\quad=x_0^{3k}Y_W^U(z_2)\big(1\otimes Y_W^V(x_2)\big)\big(Y(x_0)\otimes Y(x_0)\otimes 1\big)\\
&\qquad\cdot R^{23}(-x_0)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)\big|_{z_2=x_2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&Y_R^W\big(Y_R(u\otimes v,x_0)\big(u'\otimes v'\big),x_2\big)w\\
&\quad=Y_R^W(x_2)\big(Y(x_0)\otimes Y(x_0)\otimes 1\big)R^{23}(-x_0)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)\\
&\quad=Y_W^U(z_2)\big(1\otimes Y_W^V(x_2)\big)\big(Y(x_0)\otimes Y(x_0)\otimes 1\big)R^{23}(-x_0)\big(u\otimes v\otimes u'\otimes v'\otimes w\big)\big|_{z_2=x_2}.
\end{aligned}$$

Combining the last two equations we obtain the desired weak associativity relation. Therefore, (W,Y_W^R) carries the structure of a $U\otimes_R V$-module. It is clear that the linear map Y_W^R extends both Y_W^U and Y_W^V.

Suppose that $\bar{Y}_W$ is another $U\otimes_R V$-module structure on W, which also extends both Y_W^U and Y_W^V. For $u\in U$, $v\in V$, with the assumption (2.41), by Lemma 2.1 we have

$$\bar{Y}_W\big(Y_R(u,x_0)v,x_2\big)=\big(\bar{Y}_W(u,x_1)\bar{Y}_W(v,x_2)\big)\big|_{x_1=x_2+x_0}. \tag{2.48}$$

Recall that $Y_R(u,x)v\in(U\otimes_R V)[[x]]$ and

$$u\otimes v=u_{-1}v=\lim_{x\to 0}Y_R(u,x)v.$$

Then

$$\bar{Y}_W(u\otimes v,x)=\lim_{x_0\to 0}\bar{Y}_W\big(Y_R(u,x_0)v,x\big)=\big(\bar{Y}_W(u,x_1)\bar{Y}_W(v,x)\big)\big|_{x_1=x}=Y_R^W(u\otimes v,x).$$

Thus we have $\bar{Y}_W=Y_R^W$, proving the uniqueness assertion. □

3. Twisted tensor product of quantum vertex algebras

In this section we study twisted tensor product $U \otimes_R V$ with U and V (weak) quantum vertex algebras.

First we recall the notion of weak quantum vertex algebra from [L3].

Definition 3.1. A *weak quantum vertex algebra* is a nonlocal vertex algebra V which satisfies $\mathcal{S}$*-locality* in the sense that for $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \qquad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

(finitely many) such that

$$(x_1 - x_2)^k Y(u, x_1)Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^{r} f_i(x_2 - x_1)Y\big(v^{(i)}, x_2\big)Y\big(u^{(i)}, x_1\big) \tag{3.1}$$

for some nonnegative integer k.

The following basic facts can be found in [L3]:

Proposition 3.2. *Let V be a nonlocal vertex algebra and let*

$$u, v, u^{(i)}, v^{(i)} \in V, \qquad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r).$$

Then the $\mathcal{S}$-locality relation (3.1) *is equivalent to*

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\sum_{i=1}^{r} f_i(-x_0)Y\big(v^{(i)}, x_2\big)Y\big(u^{(i)}, x_1\big) \\ & \quad = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y\big(Y(u, x_0)v, x_2\big) \end{aligned} \tag{3.2}$$

(*the $\mathcal{S}$*-Jacobi identity)*, and it is also equivalent to*

$$Y(u, x)v = e^{x\mathcal{D}} \sum_{i=1}^{r} f_i(-x)Y\big(v^{(i)}, -x\big)u^{(i)} \tag{3.3}$$

(*the $\mathcal{S}$*-skew symmetry*).*

Proposition 3.3. *Let V be a weak quantum vertex algebra and let (W, Y_W) be a module for V viewed as a nonlocal vertex algebra. Assume*

$$u, v, u^{(i)}, v^{(i)} \in V, \qquad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

such that (3.3) *holds. Then*

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u,x_1)Y_W(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\sum_{i=1}^{r}f_i(-x_0)Y_W\big(v^{(i)},x_2\big)Y_W\big(u^{(i)},x_1\big)$$
$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_W\big(Y(u,x_0)v,x_2\big).$$

Remark 3.4. From definition, a nonlocal vertex algebra V is a weak quantum vertex algebra if and only if there exists a linear map

$$\mathcal{S}(x): V\otimes V\to V\otimes V\otimes\mathbb{C}((x))$$

satisfying the condition that for $u,v\in V$, there exists $k\in\mathbb{N}$ such that

$$\begin{aligned}&(x_1-x_2)^kY(x_1)\big(1\otimes Y(x_2)\big)(u\otimes v\otimes w)\\&\quad=(x_1-x_2)^kY(x_2)\big(1\otimes Y(x_1)\big)\mathcal{S}^{12}(x_2-x_1)(v\otimes u\otimes w)\end{aligned}\tag{3.4}$$

for every $w\in V$, or equivalently,

$$Y(x)(u\otimes v)=e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v\otimes u).\tag{3.5}$$

A *rational quantum Yang–Baxter operator* on a vector space U is a linear operator

$$\mathcal{S}(x): U\otimes U\to U\otimes U\otimes\mathbb{C}((x))$$

satisfying the quantum Yang–Baxter equation

$$\mathcal{S}^{12}(x)\mathcal{S}^{13}(x+z)\mathcal{S}^{23}(z)=\mathcal{S}^{23}(z)\mathcal{S}^{13}(x+z)\mathcal{S}^{12}(x).$$

It is said to be *unitary* if

$$\mathcal{S}(x)\mathcal{S}^{21}(-x)=1,$$

where $\mathcal{S}^{21}(x)=\sigma\mathcal{S}(x)\sigma$ with σ denoting the flip operator on $U\otimes U$.

Definition 3.5. A *quantum vertex algebra* is a weak quantum vertex algebra V equipped with a unitary rational quantum Yang–Baxter operator $\mathcal{S}(x)$ on V, satisfying

$$\mathcal{S}(x)(\mathbf{1}\otimes v)=\mathbf{1}\otimes v\quad\text{for } v\in V,\tag{3.6}$$
$$\big[\mathcal{D}\otimes 1,\mathcal{S}(x)\big]=-\frac{d}{dx}\mathcal{S}(x),\tag{3.7}$$
$$Y(u,x)v=e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v\otimes u)\quad\text{for } u,v\in V,\tag{3.8}$$
$$\mathcal{S}(x_1)\big(Y(x_2)\otimes 1\big)=\big(Y(x_2)\otimes 1\big)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1+x_2).\tag{3.9}$$

We denote a quantum vertex algebra by a pair $(V,\mathcal{S})$.

Note that this very notion is a slight modification of the same named notion in [L3] and [L4] with extra axioms (3.6) and (3.9).

The following are some axiomatic results:

Lemma 3.6. *Let V be a nonlocal vertex algebra and let $\mathcal{S}(x)$ be a unitary rational quantum Yang–Baxter equation on V. Then* (3.6), (3.7), *and* (3.9) *are equivalent to*

$$\mathcal{S}(x)(v \otimes \mathbf{1}) = v \otimes \mathbf{1} \quad \text{for } v \in V, \tag{3.10}$$

$$\left[1 \otimes \mathcal{D}, \mathcal{S}^{-1}(x)\right] = \frac{d}{dx}\mathcal{S}^{-1}(x), \tag{3.11}$$

$$\mathcal{S}(x_1)\big(1 \otimes Y(x_2)\big) = \big(1 \otimes Y(x_2)\big)\mathcal{S}^{12}(x_1 - x_2)\mathcal{S}^{13}(x_1), \tag{3.12}$$

respectively.

Proof. By unitarity we have $\mathcal{S}^{-1}(x) = \mathcal{S}^{21}(-x) = \sigma\mathcal{S}(-x)\sigma$. For $v \in V$, we have

$$\mathcal{S}(x)(v \otimes \mathbf{1}) = \sigma\sigma\mathcal{S}(x)\sigma(\mathbf{1} \otimes v) = \sigma\mathcal{S}^{-1}(-x)(\mathbf{1} \otimes v),$$
$$\mathcal{S}(x)(\mathbf{1} \otimes v) = \sigma\sigma\mathcal{S}(x)\sigma(v \otimes \mathbf{1}) = \sigma\mathcal{S}^{-1}(-x)(v \otimes \mathbf{1}).$$

It follows that $\mathcal{S}(x)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v$ if and only if $\mathcal{S}(x)(v \otimes \mathbf{1}) = v \otimes \mathbf{1}$.

Assuming (3.7), that is $[\mathcal{D} \otimes 1, \mathcal{S}(x)] = -\frac{d}{dx}\mathcal{S}(x)$, we have

$$\left[1 \otimes \mathcal{D}, \mathcal{S}^{-1}(x)\right] = \sigma\left[\mathcal{D} \otimes 1, \sigma\mathcal{S}^{-1}(x)\sigma\right]\sigma = \sigma\left[\mathcal{D} \otimes 1, \mathcal{S}(-x)\right]\sigma = \frac{d}{dx}\mathcal{S}^{-1}(x).$$

Similarly, assuming $[1 \otimes \mathcal{D}, \mathcal{S}^{-1}(x)] = \frac{d}{dx}\mathcal{S}^{-1}(x)$ we have

$$\left[\mathcal{D} \otimes 1, \mathcal{S}(x)\right] = \sigma\left[1 \otimes \mathcal{D}, \sigma\mathcal{S}(x)\sigma\right]\sigma = \sigma\left[1 \otimes \mathcal{D}, \mathcal{S}^{-1}(-x)\right]\sigma = -\frac{d}{dx}\mathcal{S}(x).$$

Note that (3.9) amounts to

$$\mathcal{S}^{-1}(x_1)\big(Y(x_2) \otimes 1\big) = \big(Y(x_2) \otimes 1\big)\big(\mathcal{S}^{-1}\big)^{13}(x_1 + x_2)\big(\mathcal{S}^{-1}\big)^{23}(x_1),$$

which is

$$\sigma\mathcal{S}(-x_1)\sigma\big(Y(x_2) \otimes 1\big) = \big(Y(x_2) \otimes 1\big)\sigma^{13}\mathcal{S}^{13}(-x_1 - x_2)\sigma^{13}\sigma^{23}\mathcal{S}^{23}(-x_1)\sigma^{23}.$$

The latter amounts to

$$\mathcal{S}(-x_1)\sigma\big(Y(x_2) \otimes 1\big) = \sigma\big(Y(x_2) \otimes 1\big)\sigma^{13}\mathcal{S}^{13}(-x_1 - x_2)\sigma^{13}\sigma^{23}\mathcal{S}^{23}(-x_1)\sigma^{23}. \tag{3.13}$$

As

$$\sigma\big(Y(x) \otimes 1\big) = \big(1 \otimes Y(x)\big)\sigma^{12}\sigma^{23},$$

(3.13) amounts to

$$\begin{aligned}\mathcal{S}(-x_1)\big(1 \otimes Y(x_2)\big) &= \big(1 \otimes Y(x_2)\big)\sigma^{12}\sigma^{23}\sigma^{13}\mathcal{S}^{13}(-x_1-x_2)\sigma^{13}\sigma^{23}\mathcal{S}^{23}(-x_1)\sigma^{12}\\ &= \big(1 \otimes Y(x_2)\big)\sigma^{23}\mathcal{S}^{13}(-x_1-x_2)\sigma^{23}\sigma^{12}\mathcal{S}^{23}(-x_1)\sigma^{12}\\ &= \big(1 \otimes Y(x_2)\big)\mathcal{S}^{12}(-x_1-x_2)\mathcal{S}^{13}(-x_1),\end{aligned}$$

which is a version of (3.12). □

Lemma 3.7. *Let $(V, \mathcal{S})$ be a quantum vertex algebra. Set*

$$R(x) = \mathcal{S}(x)\sigma : V \otimes V \to V \otimes V \otimes \mathbb{C}((x)).$$

Then $R(x)$ is an invertible twisting operator for the ordered pair (V, V).

Proof. As $\mathcal{S}(x)$ is unitary, it is clear that $R(x)$ is invertible. From Lemma 3.6 we have

$$R(x)(\mathbf{1} \otimes u) = \mathcal{S}(x)(u \otimes \mathbf{1}) = u \otimes \mathbf{1},$$
$$R(x)(v \otimes \mathbf{1}) = \mathcal{S}(x)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v$$

for $u, v \in V$. Notice that

$$\sigma\big(Y(x) \otimes 1\big) = \big(1 \otimes Y(x)\big)\sigma^{12}\sigma^{23},$$
$$\sigma\big(1 \otimes Y(x)\big) = \big(Y(x) \otimes 1\big)\sigma^{23}\sigma^{12}.$$

Using this, (3.9), and (3.12), we obtain (2.11) and (2.10). □

The following is straightforward (cf. [EK,L3]):

Proposition 3.8. *Let V be a weak quantum vertex algebra. Assume that V is non-degenerate. Then there exists a linear map $\mathcal{S}(x) : V \otimes V \to V \otimes V \otimes \mathbb{C}((x))$, which is uniquely determined by*

$$Y(u, x)v = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u) \quad \textit{for } u, v \in V.$$

Furthermore, $(V, \mathcal{S})$ carries the structure of a quantum vertex algebra and the following relation holds

$$\big[1 \otimes \mathcal{D}, \mathcal{S}(x)\big] = \frac{d}{dx}\mathcal{S}(x). \tag{3.14}$$

In view of Proposition 3.8, the term “non-degenerate quantum vertex algebra” without referring a quantum Yang–Baxter operator is non-ambiguous.

Proposition 3.9. *Let U and V be weak quantum vertex algebras and let $R(x)$ be an invertible twisting operator for (U, V). Then $U \otimes_R V$ is a weak quantum vertex algebra.*

Proof. From Remark 3.4, there are linear maps

$$\mathcal{S}_U(x) : U \otimes U \to U \otimes U \otimes \mathbb{C}((x)) \quad \text{and} \quad \mathcal{S}_V(x) : V \otimes V \to V \otimes V \otimes \mathbb{C}((x))$$

such that for $u, u' \in U$, $v, v' \in V$,

$$Y_U(x)\big(u\otimes u'\big)=e^{x\mathcal{D}}Y_U(-x)\mathcal{S}_U(-x)\big(u'\otimes u\big),$$
$$Y_V(x)\big(v\otimes v'\big)=e^{x\mathcal{D}}Y_V(-x)\mathcal{S}_V(-x)\big(v'\otimes v\big).$$

Let $u,u'\in U,\ v,v'\in V$. Using (2.22) we get

$$\begin{aligned}
&Y_R(u\otimes v,x)\big(u'\otimes v'\big)\\
&\quad=\big(Y_U(x)\otimes Y_V(x)\big)R^{23}(-x)\big(u\otimes v\otimes u'\otimes v'\big)\\
&\quad=\big(e^{x\mathcal{D}}Y_U(-x)\otimes e^{x\mathcal{D}}Y_V(-x)\big)\mathcal{S}_U^{12}(-x)\sigma^{12}\mathcal{S}_V^{34}(-x)\sigma^{34}R^{23}(-x)\big(u\otimes v\otimes u'\otimes v'\big)\\
&\quad=e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}\big(Y_U(-x)\otimes Y_V(-x)\big)\mathcal{S}_U^{12}(-x)\sigma^{12}\mathcal{S}_V^{34}(-x)\sigma^{34}R^{23}(-x)\big(u\otimes v\otimes u'\otimes v'\big)\\
&\quad=e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}Y_R(-x)\big(R^{-1}\big)^{23}(-x)\mathcal{S}_U^{12}(-x)\sigma^{12}\mathcal{S}_V^{34}(-x)\sigma^{34}R^{23}(-x)\big(u\otimes v\otimes u'\otimes v'\big).
\end{aligned}$$

Setting

$$\mathcal{S}_R(x)=\big(R^{-1}\big)^{23}(x)\mathcal{S}_U^{12}(x)\sigma^{12}\mathcal{S}_V^{34}(x)\sigma^{34}R^{23}(x)\sigma^{13}\sigma^{24}, \tag{3.15}$$

a linear map from $(U\otimes_R V)\otimes(U\otimes_R V)$ to $(U\otimes_R V)\otimes(U\otimes_R V)\otimes\mathbb{C}((x))$, we have

$$Y_R(u\otimes v,x)\big(u'\otimes v'\big)=e^{x(\mathcal{D}\otimes 1+1\otimes\mathcal{D})}Y_R(-x)\mathcal{S}_R(-x)\big(u'\otimes v'\otimes u\otimes v\big). \tag{3.16}$$

By Remark 3.4 or by Proposition 3.2, $U\otimes_R V$ is a weak quantum vertex algebra. □

Furthermore, we have:

Proposition 3.10. *Let U and V be weak quantum vertex algebras and let R be an invertible twisting operator. Assume that U as a U-module and V as a V-module are irreducible and of countable dimension over $\mathbb{C}$. Then $U\otimes_R V$ as a $U\otimes_R V$-module is irreducible. Furthermore, $U\otimes_R V$ is a non-degenerate quantum vertex algebra.*

Proof. Recall from (2.17) that

$$Y_R(u,x)\big(u'\otimes v\big)=Y(u,x)u'\otimes v\quad\text{for } u,u'\in U,\ v\in V.$$

It follows that for any subspace A of V, $U\otimes A$ is a U-submodule of $U\otimes_R V$. As U is an irreducible U-module of countable dimension over $\mathbb{C}$, we have $\mathrm{Hom}_U(U,U)=\mathbb{C}$. Let P be any $U\otimes_R V$-submodule of $U\otimes_R V$. Using the U-module structure we get $P=U\otimes A$ for some vector space A of V. Since

$$\mathbf{1}\otimes Y(v,x)a=Y_R(v,x)(\mathbf{1}\otimes a)\in(U\otimes A)\big[\big[x,x^{-1}\big]\big]$$

for $v\in V$, $a\in A\subset V$, we see that A is a V-submodule. This proves that each $U\otimes_R V$-submodule of $U\otimes_R V$ is of the form $U\otimes A$ where A is a V-submodule of V. Since V is an irreducible V-module, $U\otimes_R V$ as a $U\otimes_R V$-module is irreducible. It follows from Theorem 3.9 of [L4] that $U\otimes_R V$ is non-degenerate. On the other hand, by Proposition 3.9, $U\otimes_R V$ is a weak quantum vertex algebra. Therefore, $U\otimes_R V$ is a non-degenerate quantum vertex algebra. □

4. Smash product of nonlocal vertex algebras

In this section, we first slightly generalize the smash product construction of nonlocal vertex algebras, established in [L5], and we then prove that every smash product is a twisted tensor product with respect to a canonical twisting operator.

We begin by recalling from [L5] the basic notions and the smash product construction. For convenience, we first recall some necessary classical notions. A *coalgebra* is a vector space C (over $\mathbb{C}$) equipped with linear maps

$$\Delta : C \to C \otimes C \quad \text{and} \quad \varepsilon : C \to \mathbb{C},$$

satisfying

$$(1 \otimes \Delta)\Delta(b) = (\Delta \otimes 1)\Delta(b),$$
$$(\varepsilon \otimes 1)\Delta(b) = 1 \otimes b, \qquad (1 \otimes \varepsilon)\Delta(b) = b \otimes 1$$

for $b \in C$. For a coalgebra C, a C-*comodule* is a vector space V equipped with a linear map $\rho : V \to C \otimes V$ such that

$$(1 \otimes \rho)\rho = (\Delta \otimes 1)\rho, \tag{4.1}$$
$$(\varepsilon \otimes 1)\rho(v) = \mathbf{1} \otimes v \quad \text{for } v \in V. \tag{4.2}$$

A *nonlocal vertex bialgebra* is a nonlocal vertex algebra H equipped with a classical coalgebra structure (Δ, ε) such that both Δ and ε are homomorphisms of nonlocal vertex algebras. That is,

$$\varepsilon(\mathbf{1}) = 1, \qquad \varepsilon\big(Y(h,x)h'\big) = \varepsilon(h)\varepsilon\big(h'\big), \tag{4.3}$$
$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \qquad \Delta\big(Y(h,x)h'\big) = Y\big(\Delta(h),x\big)\Delta\big(h'\big) \quad \text{for } h,h' \in H. \tag{4.4}$$

A *nonlocal vertex H-module-algebra* is a nonlocal vertex algebra V equipped with a module structure for H viewed as a nonlocal vertex algebra such that

$$Y(h,x)v \in V \otimes \mathbb{C}((x)) \big(\subset V((x))\big), \tag{4.5}$$
$$Y(h,x)\mathbf{1} = \varepsilon(h)\mathbf{1}, \tag{4.6}$$
$$Y(h,x)Y(u,z)v = Y\big(Y\big(h^1, x-z\big)u, z\big)Y\big(h^2, x\big)v \tag{4.7}$$

for $h \in H$, $u, v \in V$, where $\Delta(h) = h^1 \otimes h^2$ in the Sweedler notation. Notice that if V is infinite-dimensional, which is true most of the time, $V \otimes \mathbb{C}((x)) \neq V((x))$.

The following is a simple fact that we shall need later:

Lemma 4.1. *Let H be a nonlocal vertex bialgebra and let V be a nonlocal vertex H-module-algebra. Then*

$$Y(h,z+x)Y\big(h',z\big)v = Y\big(Y(h,x)h',z\big)v \tag{4.8}$$

for $h, h' \in H$, $v \in V$.

Proof. Let $h, h' \in H$, $v \in V$. There exists a nonnegative integer l such that

$$(x+z)^l Y(h, x+z) Y\big(h', z\big) v = (x+z)^l Y\big(Y(h, x) h', z\big) v.$$

(Note that $Y(h, x+z) Y(h', z) v$ exists in $V((x))((z))$.) As $Y(h', z) v \in V \otimes \mathbb{C}((z))$, we have

$$Y(h, x_1) Y\big(h', z\big) v \in V((x_1)) \otimes \mathbb{C}((z)).$$

Then $Y(h, z+x) Y(h', z) v$ exists in $V((z))[[x]]$. Replace l with a large one if necessary, so that

$$x_1^l Y(h, x_1) Y\big(h', z\big) v \in V[[x_1]] \otimes \mathbb{C}((z)).$$

Because of this we have

$$(x+z)^l Y(h, x+z) Y\big(h', z\big) v = (z+x)^l Y(h, z+x) Y\big(h', z\big) v.$$

Then

$$(z+x)^l Y(h, z+x) Y\big(h', z\big) v = (x+z)^l Y\big(Y(h, x) h', z\big) v.$$

Multiplying both sides by $(z+x)^{-l}$ $(\in \mathbb{C}((z))[[x]])$, we obtain the desired relation. $\square$

Definition 4.2. Let H be a nonlocal vertex bialgebra. A *nonlocal vertex H-comodule-algebra* is a nonlocal vertex algebra V equipped with a (left) comodule structure

$$\rho : V \to H \otimes V$$

for H viewed as a coalgebra such that ρ is a homomorphism of nonlocal vertex algebras, that is,

$$\rho(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \tag{4.9}$$

$$\rho\big(Y(v, x) v'\big) = \big(Y(x) \otimes Y(x)\big) \sigma^{23}\big(\rho(v) \otimes \rho\big(v'\big)\big) \quad \text{for } v, v' \in V. \tag{4.10}$$

We have:

Proposition 4.3. *Let H be a nonlocal vertex bialgebra, let U be a nonlocal vertex* (*left*) *H-module-algebra, and let V be a nonlocal vertex* (*left*) *H-comodule-algebra. Define a linear map*

$$R(x) : V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$$

by

$$R(x)(v \otimes u) = Y\big(b^1(v), -x\big) u \otimes v^2 \quad \textit{for } v \in V,\ u \in U.$$

Then $R(x)$ is a twisting operator for the pair (U, V).

Proof. Let $u \in U$, $v \in V$. As $\rho(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $(\varepsilon \otimes 1)\rho(v) = \mathbf{1} \otimes v$, we have

$$R(x)(\mathbf{1} \otimes u) = Y(\mathbf{1}, -x)u \otimes \mathbf{1} = u \otimes \mathbf{1},$$
$$R(x)(v \otimes \mathbf{1}) = Y\big(b^1(v), -x\big)\mathbf{1} \otimes v^2 = \varepsilon\big(b^1(v)\big)\mathbf{1} \otimes v^2 = \mathbf{1} \otimes v.$$

On one hand, using (4.10) we have

$$\begin{aligned} R(z)\big(Y(x) \otimes 1\big)\big(v \otimes v' \otimes u\big) &= R(z)\big(Y(v,x)v' \otimes u\big) \\ &= Y\big(b^1\big(Y(v,x)v'\big), -z\big)u \otimes \big(Y(v,x)v'\big)^2 \\ &= Y\big(Y\big(b^1(v),x\big)b^1\big(v'\big), -z\big)u \otimes Y\big(v^2,x\big)v'^2, \end{aligned}$$

noticing that (4.10) gives

$$b^1\big(Y(v,x)v'\big) \otimes \big(Y(v,x)v'\big)^2 = Y\big(b^1(v),x\big)b^1\big(v'\big) \otimes Y\big(v^2,x\big)v'^2. \tag{4.11}$$

On the other hand, using Lemma 4.1 we get

$$\begin{aligned} &\big(1 \otimes Y(x)\big)R^{12}(z-x)R^{23}(z)\big(v \otimes v' \otimes u\big) \\ &\quad = \big(1 \otimes Y(x)\big)R^{12}(z-x)\big(v \otimes Y\big(b^1\big(v'\big), -z\big)u \otimes v'^2\big) \\ &\quad = \big(1 \otimes Y(x)\big)\big(Y\big(b^1(v), -z+x\big)Y\big(b^1\big(v'\big), -z\big)u \otimes v^2 \otimes v'^2\big) \\ &\quad = Y\big(b^1(v), -z+x\big)Y\big(b^1\big(v'\big), -z\big)u \otimes Y\big(v^2,x\big)v'^2 \\ &\quad = Y\big(Y\big(b^1(v),x\big)b^1\big(v'\big), -z\big)u \otimes Y\big(v^2,x\big)v'^2. \end{aligned}$$

Consequently, we have

$$R(z)\big(Y(x) \otimes 1\big)\big(v \otimes v' \otimes u\big) = \big(1 \otimes Y(x)\big)R^{12}(z-x)R^{23}(z)\big(v \otimes v' \otimes u\big).$$

Similarly, we have

$$\begin{aligned} R(z)\big(1 \otimes Y(x)\big)\big(v \otimes u \otimes u'\big) &= R(z)\big(v \otimes Y(u,x)u'\big) \\ &= Y\big(b^1(v), -z\big)Y(u,x)u' \otimes v^2 \\ &= Y\big(Y\big(b^1(v)^1, -z-x\big)u, x\big)Y\big(b^1(v)^2, -z\big)u' \otimes v^2, \end{aligned}$$

while

$$\begin{aligned} &\big(Y(x) \otimes 1\big)R^{23}(z)R^{12}(z+x)\big(v \otimes u \otimes u'\big) \\ &\quad = \big(Y(x) \otimes 1\big)R^{23}(z)\big(Y\big(b^1(v), -z-x\big)u \otimes v^2 \otimes u'\big) \\ &\quad = \big(Y(x) \otimes 1\big)\big(Y\big(b^1(v), -z-x\big)u \otimes Y\big(b^1\big(v^2\big), -z\big)u' \otimes v^{22}\big) \\ &\quad = Y\big(Y\big(b^1(v), -z-x\big)u, x\big)Y\big(b^1\big(v^2\big), -z\big)u' \otimes v^{22}. \end{aligned}$$

As

$$b^1(v)^1 \otimes b^1(v)^2 \otimes v^2 = b^1(v) \otimes b^1\big(v^2\big) \otimes v^{22}$$

by (4.1), we have

$$R(z)\big(1 \otimes Y(x)\big)\big(v \otimes u \otimes u'\big) = \big(Y(x) \otimes 1\big)R^{23}(z)R^{12}(z+x)\big(v \otimes u \otimes u'\big).$$

This proves that $R(x)$ is a twisting operator for the ordered pair (U, V). □

The following follows immediately from the definition of $U \otimes_R V$ and Theorem 2.4:

Corollary 4.4. *Let H, U, V be given as in Proposition* 4.3. *For $u, u' \in U$, $v, v' \in V$, define*

$$Y_\sharp(u \otimes v, x)\big(u' \otimes v'\big) = Y(u, x)Y\big(b^1(v), x\big)u' \otimes Y\big(v^2, x\big)v', \tag{4.12}$$

where $\rho(v) = b^1(v) \otimes v^2$. Then $(U \otimes V, Y_\sharp, \mathbf{1} \otimes \mathbf{1})$ carries the structure of a nonlocal vertex algebra, which we denote by $U \sharp V$. Furthermore, we have $U \sharp V = U \otimes_R V$.

Notice that a nonlocal vertex bialgebra H itself is a nonlocal vertex H-comodule-algebra with $\rho = \Delta$. Taking $V = H$ in Corollary 4.4, we obtain the smash product nonlocal vertex algebra $U \sharp H$, which was studied in [L5] (cf. [KL]). In [L5], some interesting examples, related to the vertex operator algebras associated to even lattices and those associated to infinite-dimensional Heisenberg algebras, were given.

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